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Evaluation of eigenfunctions from compound matrix variables in non-linear elasticity – II. Sixth order systems

D.M. Haughton

Department of Mathematics, University of Glasgow, University Avenue, Glasgow G12 8QW, UK

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ABSTRACT

We show how the compound matrix method can be extended to give eigenfunctions as well as eigenvalues to bifurcation problems in non-linear elasticity. The non-trivial boundary conditions create some difficulties and we find that sixth order systems for elasticity problems will require a shooting method for two functions of two unknown parameters over and above the calculations required for comparable problems in fluids.

1. Introduction

In the companion paper [1] it was shown how the compound matrix method could be extended to give eigenfunctions as well as eigenvalues for fourth order problems in solid mechanics and in particular bifurcation problems in non-linear elasticity. The method had previously been established for fourth order fluid mechanics problems by Ng and Reid [2] and Straughan and Walker [3]. To extend the methods of [2,3] to solid mechanics problems we essentially had to modify the approach to the boundary conditions. In [1] we also gave a different proof that the compound matrix eigenfunction equations will in fact give a solution to the original problem. Ng and Reid [4] have considered the sixth order case for fluids and we again find that a different approach to the boundary conditions is required for problems in solid mechanics. In this paper we show how the proofs given in [1] can be extended to sixth order problems. This is the general case for bifurcation problems in unconstrained elasticity where the incremental equations give three simultaneous equations involving the second derivatives of the three components of the incremental displacements. See Ogden [5], for example. For incompressible materials the problem formulation is slightly different. We have three incremental equilibrium equations involving the second derivatives of the three incremental displacements but these equations also involve the first derivative of the incremental (arbitrary hydrostatic) pressure. To compensate for this we have an additional first order equation that arises from the incompressibility condition (the trace of the incremental displacement gradient tensor will be zero). In most, if not all, cases the incompressibility condition can be used to eliminate one of the displacements leaving us with three equations for the two

E-mail address: dmh@maths.gla.ac.uk

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remaining displacements and the pressure. Typically this will result in one third order equation, one second order and a first order equation for the hydrostatic incremental pressure. Sometimes it is possible to eliminate the hydrostatic pressure and in these cases we are often left with higher order equations for one or two displacement components. Ogden [5] has some examples of this type of problem. The starting point that we adopt can therefore be used for bifurcation problems in unconstrained non-linear elasticity but will require some minor and obvious modifications for incompressible problems.

In Section 2 we briefly describe the basic compound matrix method [4] for such a system of three simultaneous second order equations with three first order boundary conditions applied at two points. We then apply the eigenfunction method described in [1–4]. First we give a direct proof that the solution to the eigenfunction equation is also a solution of the original problem. We then show that the boundary conditions at one end of the range are automatically satisfied. At the opposite end of the range one unknown incremental displacement (an unknown dependent variable) can be chosen to normalise the solution, since we are dealing with a homogeneous problem. We show that we have to determine the initial values of the two other unknown variables to ensure that the remaining three boundary conditions are satisfied. We use Newton's method to do this.

Following this general analysis we look at a specific non-trivial example taken from [6]. This allows us to compare exact solutions with the compound matrix method and one other numerical approach.

2. Compound matrix method

We start by giving a brief description of the compound matrix method to determine an unknown parameter (eigenvalue) described by a sixth order system. A more detailed derivation of the equations for the fourth order case can be found in [1]. We remark here that our generalised eigenvalue problem can eventually be regarded as solving $\det(A(\lambda)) = 0$ for the parameter λ. In our problem all of the entries in the matrix A will be non-linear functions of the parameter. The compound matrix method avoids the calculation of the matrix A so the derivation of the corresponding eigenvector is not straightforward. Here we consider three second order equations for $f(x)$, $g(x)$ and $h(x)$ in the form

$$f'' = \alpha_1 f + \alpha_2 f' + \alpha_3 g + \alpha_4 g' + \alpha_5 h + \alpha_6 h', \tag{1}$$
$$g'' = \beta_1 f + \beta_2 f' + \beta_3 g + \beta_4 g' + \beta_5 h + \beta_6 h', \tag{2}$$
$$h'' = \gamma_1 f + \gamma_2 f' + \gamma_3 g + \gamma_4 g' + \gamma_5 h + \gamma_6 h', \tag{3}$$

where the prime denotes differentiation with respect to x and the coefficients α_i, β_i and γ_i, $i = 1,\ldots,6$, will depend on the parameter λ, say, that we are looking for and in general on x. We also have boundary conditions

$$a_1 f' + a_2 f + a_3 g + a_4 h = 0, \quad x = a, \tag{4}$$
$$b_1 g' + b_2 f + b_3 g + b_4 h = 0, \quad x = a, \tag{5}$$
$$c_1 h' + c_2 f + c_3 g + c_4 h = 0, \quad x = a \tag{6}$$

and

$$p_1 f' + p_2 f + p_3 g + p_4 h = 0, \quad x = b, \tag{7}$$
$$q_1 g' + q_2 f + q_3 g + q_4 h = 0, \quad x = b, \tag{8}$$
$$r_1 h' + r_2 f + r_3 g + r_4 h = 0, \quad x = b. \tag{9}$$

The coefficients in the boundary conditions will also depend on the parameter λ, as will a and b.

We suppose that, in principle, Eqs. (1)–(3) are solved three times with three linearly independent initial conditions (at $x = a$) which ensure that the boundary conditions (4)–(6) are satisfied. The three solutions thus obtained are labelled f^i, g^i and h^i, $i = 1, 2, 3$. The full solution can then be written

$$\begin{aligned} f &= C_1 f^1 + C_2 f^2 + C_3 f^3, \\ g &= C_1 g^1 + C_2 g^2 + C_3 g^3, \\ h &= C_1 h^1 + C_2 h^2 + C_3 h^3, \end{aligned} \tag{10}$$

where C_1, C_2 and C_3 arbitrary constants.

We now introduce twenty new compound matrix variables $\phi_i(x)$, $i = 1,\ldots,20$, defined by 3×3 determinants. If we introduce the notation

$$(u, v, w) = \begin{vmatrix} u^1 & u^2 & u^3 \\ v^1 & v^2 & v^3 \\ w^1 & w^2 & w^3 \end{vmatrix}, \tag{11}$$

then the compound matrix variables are given in the Appendix A (35). We now differentiate (35) and use (1)–(3) with f, g and h replaced with f^1, etc. as required, to obtain the compound matrix differential equations which are also listed explicitly in Appendix A (36).

Now using the initial conditions (4)–(6), arbitrarily normalising the solution by setting $\phi_6(a) = 1$ and assuming that $a_1(a) \neq 0$, $b_1(a) \neq 0$ and $c_1(a) \neq 0$ we have the initial conditions (Appendix A (37)) for the compound matrix variables ϕ_i. We note that when one or more of a_1, b_1,c_1 is zero at $x = a$ we can still find suitable initial conditions.

It remains to ensure that the boundary conditions at $x = b$ are satisfied. We take the solutions (10) and substitute them into the boundary conditions (7)–(9). We then require the coefficient matrix for the constants C_1, C_2, C_3 to be singular for the existence of non-trivial solutions. This then leads to the requirement that a 3×3 determinant is zero. This 3×3 determinant can be written in terms of ϕ_i's and setting this to be zero gives our target condition (Appendix A (38)).

3. Compound matrix eigenfunction

Now suppose that we have found a critical value of our parameter λ so that (36) with (37) integrate to give (38) at $x = b$. We can then arrange to obtain values of $\phi_i(x)$ for any $x \in (a,b)$. If we differentiate the formal solution (10) we have

$$
\begin{aligned}
f' &= C_1 f^{1'} + C_2 f^{2'} + C_3 f^{3'},\\
g' &= C_1 g^{1'} + C_2 g^{2'} + C_3 g^{3'},\\
h' &= C_1 h^{1'} + C_2 h^{2'} + C_3 h^{3'}.
\end{aligned}
\tag{12}
$$

Now solving (10) for the constants C_1, C_2 and C_3 in terms of f, g, h and f^i, etc., then substituting these expressions for the constants back into (12) we have

$$
\begin{aligned}
\phi_6 f' &= \phi_{12} f + \phi_3 g - \phi_1 h,\\
\phi_6 g' &= -\phi_{17} f + \phi_8 g - \phi_5 h,\\
\phi_6 h' &= -\phi_{19} f - \phi_{10} g + \phi_7 h,
\end{aligned}
\tag{13}
$$

having multiplied by ϕ_6 which we assume is non-zero throughout the range $x \in (a,b)$. We recall that we arbitrarily set $\phi_6(a) = 1$. Eqs. (13) are then the equations that we use to determine f, g and h along with suitable initial conditions for $f(a)$, $g(a)$ and $h(a)$ that are yet to be found.

If we consider (13) at $x = a$ we can substitute the initial conditions for the ϕ_i's from (37) and we see that the initial conditions (4)–(6) are automatically satisfied. Thus we are free to impose any initial conditions on $f(a)$, $g(a)$ and $h(a)$. To normalise the solution we set

$$f(a) = 1.$$

As we shall see below we must choose particular values for $g(a)$ and $h(a)$ in order that the three remaining boundary conditions (7)–(9) are satisfied.

We shall first prove that a solution to (13) with initial conditions $f(a) = 1$ and $g(a) = g_a$, $h(a) = h_a$ is also a solution to the original problem (1)–(3). We shall focus attention on the equation for f but the other equations can be dealt with in a similar way. Unfortunately some of the intermediate results require rather large expressions. The bulky algebraic manipulations were mainly done with Maple.

First we differentiate $(13)_1$ and we then use Eqs. (36) to substitute for the ϕ_i derivatives and (13) to substitute for the derivatives of f, g and h to give

$$
\begin{aligned}
\phi_6^2 f'' = {} & [-\phi_1\phi_{19} - \phi_3\phi_{17} + (\phi_6\alpha_2 + \phi_{14} + \phi_{19}\alpha_5 + \phi_{12}\alpha_1 + \phi_{13} - \phi_{17}\alpha_3)\phi_6 - \phi_{12}(\phi_8 + \phi_7)]f\\
& + [\phi_1\phi_{10} - \phi_7\phi_3 + (\phi_4 + \alpha_1\phi_3 + \alpha_4\phi_6 + \phi_8\alpha_3 - \phi_{10}\alpha_5)\phi_6]g\\
& + [\phi_1\phi_8 + \phi_3\phi_5 + (\phi_5\alpha_3 + \phi_6\alpha_6 + \phi_7\alpha_5 - \phi_2 - \phi_1\alpha_1)\phi_6]h.
\end{aligned}
\tag{14}
$$

Next we subtract $(\phi_6)^2$ times the right-hand side of (1) from both sides of (14) to get

$$
\phi_6^2 L_1(f,g,h) = (\phi_1\phi_{19} + \phi_3\phi_{17} - \phi_6(\phi_{13} + \phi_{14}) + \phi_{12}(\phi_8 + \phi_7))f + (\phi_3\phi_7 - \phi_1\phi_{10} - \phi_6\phi_4)g + (\phi_6\phi_2 - \phi_1\phi_8 - \phi_3\phi_5)h,
\tag{15}
$$

where $L_1(f,g,h)$ is the differential equation (1) and we have again used (13). We now recognise from (35) that we have the following identities:

$$\phi_6\phi_{13} - \phi_1\phi_{19} - \phi_7\phi_{12} \equiv 0, \quad \phi_6\phi_{14} - \phi_3\phi_{17} - \phi_8\phi_{12} \equiv 0, \tag{16}$$

$$\phi_3\phi_7 - \phi_1\phi_{10} - \phi_6\phi_4 \equiv 0, \quad \phi_6\phi_2 - \phi_1\phi_8 - \phi_3\phi_5 \equiv 0. \tag{17}$$

Hence the right-hand side of (15) is identically zero and the original Eq. (1) is satisfied. In a very similar way we can show that the other two equilibrium Eqs. (2) and (3) are also satisfied. See Ng and Reid [4] for a general discussion of identities arising from a sixth order system.

We shall now assume that the initial conditions $g(a) = g_a$, $h(a) = h_a$ have been chosen so that the boundary conditions (8) and (9) at $x = b$ are satisfied. It remains to be shown that the final boundary condition (7) is also satisfied. This is an arbitrary choice; we just need to assume any two of the boundary conditions are satisfied (with a suitable choice of g_a and h_a) and we can then prove that the third one is also satisfied. To do this we first consider (8) and write

$$\phi_6(q_1g' + q_2f + q_3g + q_4h) = q_1(-\phi_{17}f + \phi_8g - \phi_5h) + \phi_6(q_2f + q_3g + q_4h) = 0, \quad x = b, \tag{18}$$

having used (13) and so

$$(\phi_6q_2 - \phi_{17}q_1)f + (\phi_8q_1 + \phi_6q_3)g + (\phi_6q_4 - \phi_5q_1)h = 0. \tag{19}$$

Similarly from (9) we have

$$(\phi_6r_2 - \phi_{19}r_1)f + (\phi_6r_3 - \phi_{10}r_1)g + (\phi_6r_4 + \phi_7r_1)h = 0. \tag{20}$$

Now we consider the left-hand side of (7) and we show that it is necessarily zero. Let ψ be given by

$$\psi = \phi_6(p_1f' + p_2f + p_3g + p_4h) = (p_1\phi_{12} + p_2\phi_6)f + (p_1\phi_3 + p_3\phi_6)g + (p_4\phi_6 - p_1\phi_1)h, \tag{21}$$

having used $(13)_1$. Now we solve Eqs. (19) and (20) for $g(b)$ and $h(b)$ in terms of $f(b)$ and substitute into the right-hand side of (21). After a little rearranging and making use of the identities (16) and (17) and the further identities given in Appendix B (39)–(46), together with the target condition (38) allows us to show that ψ is identically zero.

To summarise, the method for determining f, g and h is to choose a value for $g(a) = g_a$ and $h(a) = h_a$ with $f(a) = 1$, then integrate equations (13) to $x = b$. We then adjust the value of (g_a, h_a) to ensure that any two of (7)–(9) are satisfied. It follows from the above that the final boundary condition at $x = b$ will also hold. From (13) the boundary conditions (4)–(6) will be satisfied and so we have a solution to the original problem.

4. Example: An elastic tube under axial compression

The purpose of this section is to verify that the compound matrix method does work and to compare the results it gives with one other possible numerical approach. The example that we consider is taken from [6] and has an exact solution in terms of Bessel functions. The special case of a zero mode number where the problem reduces to one of fourth order was considered as an example in [1].

We suppose that the cylindrical tube is composed of a compressible Neo-Hookean material with a strain-energy function of the specific form

$$W = \mu(I_1 - 3)/2 - (\kappa + \mu/3)\log(J) - (2/3\mu - \kappa)(J - 1), \tag{22}$$

where $I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$ and $J = \lambda_1\lambda_2\lambda_3$ in terms of the principal stretches λ_i, $i = 1, \ldots, 3$. We take the shear modulus $\mu = 1$, to normalise the equations. The bulk modulus for a very compressible material is taken to be $\kappa = 5$. The undeformed tube has a length to outer radius ratio $L/B = 7$ with external and internal radii $B = 1$ and $A = 2/3$. There is also an associated integer mode number m that we shall take as $m > 0$ for the moment. In this case the incremental equilibrium equations can be written

$$f'' + \frac{f'}{r} - \frac{(208\lambda + 441r^2 + 48m^2 + 96 + 1911\lambda r^2)}{16r^2(6 + 13\lambda)}f + \frac{m(3 + 13\lambda)}{r(6 + 13\lambda)}g' - \frac{m(13\lambda + 9)}{r^2(6 + 13\lambda)}g + \frac{7(3 + 13\lambda)}{\lambda(6 + 13\lambda)}h' = 0, \tag{23}$$

$$g'' - \frac{m(3 + 13\lambda)}{3r}f' - \frac{m(13\lambda + 9)}{3r^2}f + \frac{g'}{r} - \frac{(96m^2 + 1911\lambda r^2 + 441r^2 + 208m^2\lambda + 48)}{48r^2}g - \frac{7m(3 + 13\lambda)}{3r\lambda}h = 0, \tag{24}$$

$$h'' - \frac{7(3 + 13\lambda)}{3\lambda}f' - \frac{7(3 + 13\lambda)}{3r\lambda}f - \frac{7m(3 + 13\lambda)}{3r\lambda}g + \frac{h'}{r} - \frac{(441\lambda^2r^2 + 1911\lambda^3r^2 + 48\lambda^2m^2 + 10192\lambda r^2 + 2352r^2)}{48\lambda^2r^2}h = 0, \tag{25}$$

where r is the deformed radial coordinate, with solution

$$\begin{aligned} 7k_1k_2rf(r) = {} & (7k_2I_{m+1}(\hat{k}_1)k_1r + k_2\lambda mI_m(\hat{k}_1))C_1 \\ & + (7k_1I_{m+1}(\hat{k}_2)k_2r + k_1\lambda mI_m(\hat{k}_2))C_2 \\ & + (7k_2K_{m+1}(\hat{k}_1)k_1r - k_2\lambda mK_m(\hat{k}_1))C_3 \\ & + (7k_1K_{m+1}(\hat{k}_2)k_2r - k_1\lambda mK_m(\hat{k}_2))C_4 \\ & + 7k_1k_2C_5I_m(\hat{k}_1) - 7k_1k_2C_6K_m(\hat{k}_1), \end{aligned} \tag{26}$$

$$\begin{aligned} 7\lambda mk_1k_2rg(r) = {} & -C_1m^2I_m(\hat{k}_1)\lambda^2k_2 - C_2m^2I_m(\hat{k}_2)\lambda^2k_1 \\ & + C_3m^2K_m(\hat{k}_1)\lambda^2k_2 + C_4m^2K_m(\hat{k}_2)\lambda^2k_1 \\ & + (-49k_1^2k_2I_{m+1}(\hat{k}_1)r - 7\lambda k_1k_2mI_m(\hat{k}_1))C_5 \\ & + (-49k_1^2k_2K_{m+1}(\hat{k}_1)r + 7\lambda k_1k_2mK_m(\hat{k}_1))C_6, \end{aligned} \tag{27}$$

$$\begin{aligned} 16(k_1k_2(3 + 13\lambda))h(r) = {} & (-169k_2\lambda^4 - 9k_2\lambda^2 - 78k_2\lambda^3)I_m(\hat{k}_1)C_1 \\ & + (-96k_1k_2^2 - 208k_1k_2^2\lambda + 9k_1\lambda^2 + 39k_1\lambda^3)I_m(\hat{k}_2)C_2 \\ & + (9k_2\lambda^2 + 78k_2\lambda^3 + 169k\_2\lambda^4)K_m(\hat{k}_1)C_3 \\ & + (96k_1k_2^2 + 208k_1k_2^2\lambda - 9k_1\lambda^2 - 39k_1\lambda^3)K_m(\hat{k}_2)C_4, \end{aligned} \tag{28}$$

where $C_1, \ldots, C_6$ are constants, I_m and K_m are modified Bessel functions of the first and second kind. The parameters k_1 and k_2 are given by

$$k_1 = \lambda\sqrt{3+13\lambda}/4 \tag{29}$$

and

$$k_2 = \sqrt{\frac{9\lambda^2+39\lambda^3+48+208\lambda}{16(6+13)\lambda}} \tag{30}$$

with

$$\hat{k}_1 = 7\frac{k_1 r}{\lambda}, \quad \hat{k}_2 = 7\frac{k_2 r}{\lambda}. \tag{31}$$

The boundary conditions at both $r = 2\lambda/3$ and $r = \lambda$ are

$$f' + \frac{13\lambda f}{r(6+13\lambda)} + \frac{13\lambda mg}{r(6+13\lambda)} + \frac{91h}{6+13\lambda} = 0, \tag{32}$$

$$g' - \frac{mf}{r} - \frac{g}{r} = 0, \tag{33}$$

$$h' - 7\frac{f}{\lambda} = 0. \tag{34}$$

Although this problem admits an exact solution in terms of Bessel functions we still have to evaluate the zero's of a 6×6 determinant in order to find the bifurcation parameter λ (for a given mode number m). Having found λ we compute the 6×6 coefficient matrix and find the eigenvector corresponding to the smallest eigenvalue (which, ideally, will be zero) for the constants C_i. Since we do not have a simple numerical evaluation of the exact results we cannot be sure that they will be anymore accurate than either of the two other methods that we consider. For this reason we simply present the real solutions in Tables 1 and 2. We want to compare the results obtained using the compound matrix method outlined above with the exact solution but also with numerical results obtained using the determinantal method which is described in [1]. Briefly, this method starts with the formal solution for $f(x)$, $g(x)$ and $h(x)$ in the form (10). We substitute these solutions directly into the boundary conditions (7)–(9) to obtain three homogeneous equations for the constants C_1, C_2 and C_3. For non-trivial solutions we then set the appropriate 3×3 determinant to be zero. This determines the critical value for λ. We then take the 3×3 coefficient matrix and determine its eigenvalues and eigenvectors. At least one eigenvalue will be close to zero and we take the corresponding eigenvector, suitably normalised, to give C_1, C_2, C_3 and hence $f(x)$, $g(x)$ and $h(x)$. However, the above example exposes some of the problems associated with this determinantal approach. (See, for example, Wilkinson [7] for a discussion of similar problems.) In this case the 3×3 determinant that we evaluate is a nice function of the parameter λ with a simple root. However, when we take the 3×3 matrix evaluated at the approximate parameter value we find that, in some circumstances, the matrix has two eigenvalues close to zero. These two eigenvalues have both a small real and imaginary part (the numerical method we used to determine the eigenvalues does not assume that we must have conjugate pairs). The two corresponding eigenvectors have both real and imaginary parts. However, the original problem is purely real and so we have to regard the imaginary parts of the required eigenvector as an unavoidable error. For some examples (but not those given here) we find that the erroneous imaginary parts can be of the same order of magnitude as the real solution.

Table 1
Solution obtained from the exact solution, the determinantal method and the compound matrix method: mode $m = 1$ and $f(a) = 1$

	Exact	Compound	Determinantal
λ	0.590316973329	0.590316973329	0.590317005992
$f(b)$	0.932386153975	0.932385845764	0.932386198960
$g(a)$	−0.126080114832	−0.126080061147	−0.126080119712
$g(b)$	−1.946762092791E−02	−1.946757130263E−02	−1.946761388208E−02
$h(a)$	−0.718819869939	−0.718819400911	−0.718819887913
$h(b)$	0.582312626377	0.582312215646	0.582312679521
λ	0.382951941167	0.382951941681	0.382951920416
$f(b)$	−0.691662544161	−0.692161630205	−0.691662772000
$g(a)$	−6.704159604713E−02	−6.704204974500E−02	−6.704158810140E−02
$g(b)$	−6.471979872763E−03	−6.476481438066E−03	−6.471961693906E−03
$h(a)$	−0.658031129079	−0.658028674414	−0.658031117884
$h(b)$	−0.413781587358	−0.414074065853	−0.413781678107

Table 2
Solution obtained from the exact solution, the determinantal method and the compound matrix method: mode $m = 10$ and $f(a) = 1$

	Exact	Compound	Determinantal
λ	0.332747216821	0.332747216821	0.332747958779
$f(b)$	4.81157839011	4.81157677414	4.81162422188
$g(a)$	−0.439841029508	−0.439840851607	−0.439842114712
$g(b)$	0.430575090880	0.430574915442	0.430583172698
$h(a)$	−0.586045605367	−0.586045248303	−0.586045387638
$h(b)$	3.08973725606	3.08973625616	3.08977136024
λ	0.277506389733	0.277506389809	0.277506308940
$f(b)$	−0.120372102060	−0.120667879065	−0.120382807079
$g(a)$	−0.340221101367	−0.340221624516	−0.340220916941
$g(b)$	-1.687635736993E−02	−1.691784659709E−02	−1.687697785244E−02
$h(a)$	−0.611359043726	−0.611358294635	−0.611359109623
$h(b)$	−7.391841632124E−02	−7.410000870070E−02	−7.392438679504E−02

Since the compound matrix method and the determinantal methods are very dissimilar it is difficult to make a direct comparison. For the results presented in Table 1 we have set the tolerances for the differential equation solvers in the compound matrix method to be small enough for a relative error of 5×10^{-7} and so we should expect to have around six decimal places of accuracy. Where the compound matrix or the determinantal method give spurious imaginary parts in the solutions they are ignored. In both of Tables 1 and 2 we find that there are two critical values of the parameter λ (which gives the ratio of deformed cylinder length to original cylinder length) for a given mode number m. We have shown the results for mode number $m = 1$ since this is often the critical mode number, see [6] for details. We have also included in Table 2 the results for $m = 10$ so that the effects of more severe deformations can be see. Also, although it is not obvious from the results the nature of the calculations changes from first to second root and from mode number to mode number. Sometimes the eigenvalues have a single value close to zero and sometimes a complex conjugate pair. In all cases we had to perform some preliminary calculations to find suitable starting values for $g(a)$ and $h(a)$ for the compound matrix method.

5. Concluding remarks

As we can see from Table 1 all three methods give very similar results. In particular the exact solution and the compound matrix method giving marginally closer results for the eigenvalue than the determinantal method, but this is reversed for the eigenfunctions where the compound matrix method gives slightly different results to the other two methods. For most practical purposes (graphing results say) where only two or three significant figures are required all three methods give acceptable (and equivalent) results.

Given the complexity of the compound matrix method for eigenfunctions, the amount of (numerical) work required to obtain the solutions together with potential problems in finding good starting estimates for the Newton iteration, it is perhaps appropriate to reconsider the basic determinantal method in a more favourable light. Certainly the basic determinantal method for finding the eigenvalue will always suffer from the numerical evaluation of a determinant, however there are ways round this (potential) problem. One such approach has been suggested by Amar and Goriely [8]. The eigenfunction calculation based on the determinantal method requires only the evaluation of eigenvectors of a small dimension matrix and so it is a robust calculation using well known methods. Even if this introduces erroneous imaginary parts the real solution seems to be at least as good as the compound matrix calculation, if not better. It seems difficult to avoid the conclusion that the determinantal method should be investigated with a view to improving the basic method for eigenvalues. The main problem should be to find ways of removing the numerical evaluation of the determinant without imposing an unacceptable amount of preparatory work prior to the numerical calculations. Perhaps the simplest overall approach to new problems would be to use the compound matrix method to determine the eigenvalue and then use the determinantal method for the eigenfunction.

Appendix A. The compound matrix equations

The compound matrix variables that we have used are given by

$$\begin{aligned}
&\phi_1 = (f, f', g), \quad \phi_2 = (f, f', g'), \quad \phi_3 = (f, f', h), \quad \phi_4 = (f, f', h'),\\
&\phi_5 = (f, g, g'), \quad \phi_6 = (f, g, h), \quad \phi_7 = (f, g, h'), \quad \phi_8 = (f, g', h),\\
&\phi_9 = (f, g', h'), \quad \phi_{10} = (f, h, h'), \quad \phi_{11} = (f', g, g'), \quad \phi_{12} = (f', g, h),\\
&\phi_{13} = (f', g, h'), \quad \phi_{14} = (f', g', h), \quad \phi_{15} = (f', g', h'), \quad \phi_{16} = (f', h, h'),\\
&\phi_{17} = (g, g', h), \quad \phi_{18} = (g, g', h'), \quad \phi_{19} = (g, h, h'), \quad \phi_{20} = (g', h, h').
\end{aligned} \tag{35}$$

If we differentiate (35) and use (1)–(3) we obtain the compound matrix differential equations

$$
\begin{aligned}
\phi_1' &= -\phi_1\alpha_1 + \phi_5\alpha_3 + \phi_7\alpha_5 + \phi_6\alpha_6 + \phi_2,\\
\phi_2' &= -\phi_2\alpha_1 - \phi_5\alpha_4 + \phi_9\alpha_5 + \phi_8\alpha_6 - \phi_2\beta_3 - \phi_1\beta_4 - \phi_4\beta_5 - \phi_3\beta_6,\\
\phi_3' &= -\phi_3\alpha_1 - \phi_8\alpha_3 - \phi_6\alpha_4 + \phi_{10}\alpha_5 + \phi_4,\\
\phi_4' &= -\phi_4\alpha_1 - \phi_9\alpha_3 - \phi_7\alpha_4 - \phi_{10}\alpha_6 - \phi_2\gamma_3 - \phi_1\gamma_4 - \phi_4\gamma_5 - \phi_3\gamma_6,\\
\phi_5' &= \phi_1\beta_1 - \phi_5\beta_3 - \phi_7\beta_5 - \phi_6\beta_6 + \phi_{11},\\
\phi_6' &= \phi_7 + \phi_8 + \phi_{12},\\
\phi_7' &= \phi_9 + \phi_{13} + \phi_1\gamma_1 - \phi_5\gamma_3 - \phi_7\gamma_5 - \phi_6\gamma_6,\\
\phi_8' &= -\phi_3\beta_1 + \phi_{10}\beta_5 - \phi_8\beta_3 - \phi_6\beta_4 + \phi_9 + \phi_{14},\\
\phi_9' &= -\phi_4\beta_1 - \phi_9\beta_3 - \phi_7\beta_4 - \phi_{10}\beta_6 + \phi_2\gamma_1 + \phi_5\gamma_4 - \phi_9\gamma_5 - \phi_8\gamma_6 + \phi_{15},\\
\phi_{10}' &= \phi_3\gamma_1 + \phi_8\gamma_3 + \phi_6\gamma_4 - \phi_{10}\gamma_5 + \phi_{16},\\
\phi_{11}' &= -\phi_{11}\alpha_1 - \phi_5\alpha_2 - \phi_{18}\alpha_5 - \phi_{17}\alpha_6 - \phi_1\beta_2 - \phi_{11}\beta_3 - \phi_{13}\beta_5 - \phi_{12}\beta_6,\\
\phi_{12}' &= \phi_{13} + \phi_{14} - \phi_{12}\alpha_1 + \phi_{17}\alpha_3 - \phi_{19}\alpha_5 - \phi_6\alpha_2,\\
\phi_{13}' &= -\phi_{13}\alpha_1 - \phi_7\alpha_2 + \phi_{18}\alpha_3 + \phi_{19}\alpha_6 - \phi_1\gamma_2 - \phi_{11}\gamma_3 - \phi_{13}\gamma_5 - \phi_{12}\gamma_6 + \phi_{15},\\
\phi_{14}' &= -\phi_{14}\alpha_1 - \phi_8\alpha_2 - \phi_{17}\alpha_4 - \phi_{20}\alpha_5 + \phi_3\beta_2 - \phi_{14}\beta_3 - \phi_{12}\beta_4 + \phi_{16}\beta_5 + \phi_{15},\\
\phi_{15}' &= -\phi_{15}\alpha_1 - \phi_9\alpha_2 - \phi_{18}\alpha_4 + \phi_{20}\alpha_6 + \phi_4\beta_2 - \phi_{15}\beta_3 - \phi_{13}\beta_4 - \phi_{16}\beta_6\\
&\quad - \phi_2\gamma_2 + \phi_{11}\gamma_4 - \phi_{15}\gamma_5 - \phi_{14}\gamma_6,\\
\phi_{16}' &= -\phi_{16}\alpha_1 - \phi_{10}\alpha_2 - \phi_{20}\alpha_3 - \phi_{19}\alpha_4 - \phi_3\gamma_2 + \phi_{14}\gamma_3 + \phi_{12}\gamma_4 - \phi_{16}\gamma_5,\\
\phi_{17}' &= \phi_{12}\beta_1 + \phi_6\beta_2 - \phi_{17}\beta_3 + \phi_{19}\beta_5 + \phi_{18},\\
\phi_{18}' &= \phi_{13}\beta_1 + \phi_7\beta_2 - \phi_{18}\beta_3 - \phi_{19}\beta_6 - \phi_{11}\gamma_1 - \phi_5\gamma_2 - \phi_{18}\gamma_5 - \phi_{17}\gamma_6,\\
\phi_{19}' &= -\phi_{12}\gamma_1 - \phi_6\gamma_2 + \phi_{17}\gamma_3 - \phi_{19}\gamma_5 + \phi_{20},\\
\phi_{20}' &= -\phi_{16}\beta_1 - \phi_{10}\beta_2 - \phi_{20}\beta_3 - \phi_{19}\beta_4 - \phi_{14}\gamma_1 - \phi_8\gamma_2 - \phi_{17}\gamma_4 - \phi_{20}\gamma_5.
\end{aligned}
\tag{36}
$$

If we use the initial conditions (4)–(6) in (35) we have the initial conditions for ϕ_i at $x = a$

$$
\begin{aligned}
&\phi_1 = \frac{a_4}{a_1},\quad \phi_2 = \frac{a_3b_4 - a_4b_3}{a_1b_1},\quad \phi_3 = -\frac{a_3}{a_1},\quad \phi_4 = \frac{a_3c_4 - a_4c_3}{a_1c_1},\\
&\phi_5 = -\frac{b_4}{b_1},\quad \phi_6 = 1,\quad \phi_7 = -\frac{c_4}{c_1},\quad \phi_8 = -\frac{b_3}{b_1},\quad \phi_9 = \frac{b_3c_4 - b_4c_3}{b_1c_1},\\
&\phi_{10} = \frac{c_3}{c_1},\quad \phi_{11} = \frac{a_2b_4 - b_2a_4}{a_1b_1},\quad \phi_{12} = -\frac{a_2}{a_1},\quad \phi_{13} = \frac{a_2c_4 - c_2a_4}{a_1c_1},\\
&\phi_{14} = \frac{a_2b_3 - b_2a_3}{a_1b_1},\\
&\phi_{15} = \frac{a_2(b_4c_3 - b_3c_4) + a_3(b_2c_4 - b_4c_2) + a_4(b_3c_2 - b_2c_3)}{a}{}_1 b_1c_1,\\
&\phi_{16} = \frac{c_2a_3 - a_2c_3}{a_1c_1},\quad \phi_{17} = \frac{b_2}{b_1},\quad \phi_{18} = \frac{c_2b_4 - b_2c_4}{b_1c_1},\quad \phi_{19} = -\frac{c_2}{c_1},\\
&\phi_{20} = \frac{c_2b_3 - b_2c_3}{b_1c_1}.
\end{aligned}
\tag{37}
$$

Finally, if we write the boundary conditions (7)–(9) in terms of the solutions (12) we require the determinant of the coefficient matrix for the constants C_1, C_2 and C_3 to have zero determinant. In terms of the ϕ_i's this gives us the target condition

$$
\begin{aligned}
&\{(\phi_2c_2 + \phi_{14}c_4 + \phi_{15}c_1 - \phi_{11}c_3)b_1 - (\phi_4c_1 + \phi_3c_4 + \phi_1c_3)b_2 + (\phi_{12}c_4 + \phi_1c_2 + \phi_{13}c_1)b_3 + (\phi_3c_2 - \phi_{12}c_3 + \phi_{16}c_1)b_4\}a_1\\
&\quad + \{(\phi_9c_1 - \phi_5c_3 + \phi_8c_4)b_1 + (\phi_7c_1 + \phi_6c_4)b_3 + (\phi_{10}c_1 - \phi_6c_3)b_4\}a_2 + \{(\phi_5c_2 + \phi_{17}c_4 + \phi_{18}c_1)b_1 - (\phi_7c_1 + \phi_6c_4)b_2\\
&\quad + (\phi_{19}c_1 + \phi_6c_2)b_4\}a_3 + \{-(\phi_8c_2 + \phi_{20}c_1 + \phi_{17}c_3)b_1 + (\phi_6c_3 - \phi_{10}c_1)b_2 - (\phi_{19}c_1 + \phi_6c_2)b_3\}a_4 = 0,\qquad x = b.
\end{aligned}
\tag{38}
$$

Appendix B. Identities

Below we give a list of the identities that are required to establish the results given in Section 3. They can all be verified simply by substituting for ϕ_i from (35).

$$(\phi_8\phi_{19} - \phi_{17}\phi_{10})\phi_1 + (\phi_5\phi_{19} + \phi_7\phi_{17})\phi_3 + \phi_{12}\phi_7\phi_8 + \phi_{12}\phi_5\phi_{10} - \phi_6^2\phi_{15} \equiv 0, \tag{39}$$
$$\phi_1\phi_8 - \phi_6\phi_2 + \phi_3\phi_5 \equiv 0, \tag{40}$$
$$\phi_1\phi_{17} - \phi_{12}\phi_5 + \phi_6\phi_{11} \equiv 0, \tag{41}$$
$$\phi_1\phi_{10} + \phi_6\phi_4 - \phi_3\phi_7 \equiv 0, \tag{42}$$
$$\phi_3\phi_{19} - \phi_6\phi_{16} + \phi_{12}\phi_{10} \equiv 0, \tag{43}$$
$$\phi_7\phi_8 - \phi_6\phi_9 + \phi_5\phi_{10} \equiv 0, \tag{44}$$
$$\phi_7\phi_{17} - \phi_6\phi_{18} + \phi_5\phi_{19} \equiv 0, \tag{45}$$
$$\phi_6\phi_{20} - \phi_8\phi_{19} + \phi_{17}\phi_{10} \equiv 0. \tag{46}$$

There are other independent identities involving the ϕ_i's that we have not needed to use.

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